Recall:

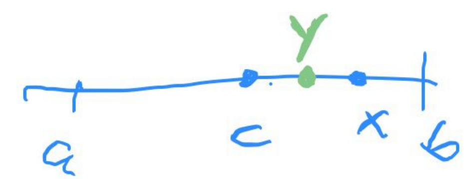
Taylor's Theorem

f defined on (a,b), $a < c < b$, $f^{(n)}(c)$ exists

let $$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$\Rightarrow$ For all $x \in (a,b)$, $x \neq c$ there exists a y between x and c such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$



Observe: Taylor series converges to $f(x)$ $\Leftrightarrow$ $R_n(x) \to 0$ for $n \to \infty$

Corollary If all derivatives of f exist in (and are bounded

(i.e. $\exists\, C$ s.t. $|f^{(n)}(y)| < C$ for all n and all y in (a,b))

$\Rightarrow$ $\lim_{n \to \infty} R_n(x) = 0$

proof $$|R_n(x)| = \left| \frac{f^{(n)}(y)}{n!}(x-c)^n \right|$$

$$\leq \frac{C}{n!}|x-c|^n$$

shown in 142A: $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ for any positive number a.

apply this for $a = |x-c|$.

$$\Rightarrow \lim_{n\to\infty} |R_n(x)| \leq \lim \frac{C}{n!}|x-c|^n = 0$$

$$\Rightarrow R_n(x) \to 0 \quad \text{for } n \to \infty$$

Examples: ① $f(x) = e^x$ $\quad c = 0$

$$\left.\begin{aligned} f'(x) &= e^x \\ \Rightarrow f^{(n)}(x) &= e^x \text{ by induction.} \end{aligned}\right\} f^{(n)}(0) = 1 \text{ for all } n.$$

$\Rightarrow$ Taylor series of $f(x)$ at $c=0$ given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

convergence? use Corollary for interval $(0,x)$ (if $x>0$)

[for $(x,0)$ if $x<0$]

if $y \in [0,x]$ $|f^{(n)}(y)| = |e^y| \le |e^x| = C$ (x fixed!)

$\Rightarrow |f^{(n)}(y)|$ bounded in $(0,x)$ $\Longrightarrow$ convergence for x.

x arbitrary $\Rightarrow$ convergence for all x.

(2) $f(x) = \sin x \qquad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$$\Rightarrow \quad f^{(n)}(x) = \begin{cases} \sin x & n = 0, 4, 8 \ldots \\ \cos x & n = 1, 5, 9 \ldots \\ -\sin x & n = 2, 6, 10, \ldots \\ -\cos x & n = 3, 7, 11, \ldots \end{cases}$$

$\underline{c=0}$

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ 1 & n = 1, 5, 9, \ldots \\ -1 & n = 3, 7, 11 \end{cases}$$

Taylor series of sine given by

$$\frac{1}{1!} x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \ldots$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}$$

convergence?

$$f^{(n)}(x) = \begin{cases} \pm\cos x \\ \pm\sin x \end{cases}$$

$$\Rightarrow |f^{(n)}(x)| \leq 1$$ for all x, for all n

$\Rightarrow$ convergence for all x!

3. $f(x) = \ln(1+x)$ ($= \log_e(1+x)$ in notations of book)

$$\left.\begin{aligned} f'(x) &= \frac{1}{1+x} = (1+x)^{-1} \\ f''(x) &= (-1)(1+x)^{-2} \\ f'''(x) &= (-1)(-2)(1+x)^{-3} \\ f^{(4)}(x) &= (-1)(-2)(-3)(1+x)^{-4} \end{aligned}\right\}$$

$$\boxed{f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}}$$

If $x=0$.

$$\boxed{f^{(n)}(0) = (-1)^{n-1}(n-1)!}$$

$\Rightarrow$ Taylor series for $\ln(1+x) = f(x)$ given by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

Observe: $f(0) = \ln(1+0) = \ln(1) = 0$

$\Rightarrow$ no constant term.

$\Rightarrow$ can start Taylor series at $k=1$.

(4) Let α be a real number, $x > 0$

Can define $x^\alpha = (e^{\ln x})^\alpha = e^{\alpha \ln x}$

$\Rightarrow$ If $\boxed{f(x) = x^\alpha}$ $(x>0)$

$\Rightarrow f'(x) = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \boxed{\alpha x^{\alpha-1} = f'(x)}$

use this differentiation rule for functions

$$f(x) = (1+x)^{\alpha}$$

$$f'(x) = \alpha(1+x)^{\alpha-1}$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

$$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$

$$\boxed{f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}}$$

$$\boxed{f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)}$$

$\Rightarrow$ Taylor series of $f(x)$ at $c=0$ given by

$$\boxed{\sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k \overset{?}{=} (1+x)^{\alpha}}$$

need to determine for which x series converges!

generalized binomial theorem

<u>next time</u>: series converges for $|x|<1$.